A CELESTIAL SPHERE MODEL IN GEOGEBRA

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Abstract

The article presents a celestial sphere model made in GeoGebra. The authors describe the mathematical tools necessary to model celestial objects using spherical coordinates and the construction of stars, circles, and meridians. They incorporate a brief description of the construction of constellations using GeoGebra through JavaScript. In addition, the article describes the modeling of the ecliptic and the positions of the Sun and the Earth to understand their apparent movements and the concept of the Zodiac.

Keywords: GeoGebra, science, celestial sphere, rotation motion, constellations

1 INTRODUCTION

Many high school education curricula incorporate the study of the Solar System and the planets as well as the main stars and constellations. This is a transversal topic for many disciplines and can be the motivation source for students of different subjects: Geography students in the study of maps, Classical Studies students due to the connection between mythology and constellations, Physics students in the study of traditional gravitation theory and the Kepler laws, Mathematics students to understand the three-dimensional representation of points either with rectangular or spherical coordinates and Arts and Crafts students, in the construction of small models.

In collaboration with other educational institutions, our teaching groups complement formal education with workshops that, having Astronomy as the linking subject among disciplines, emphasize the importance of understanding the concepts from different points of view. These workshops are mainly intended for students under 16 interested in STEM subjects (Science, Technology, Engineering, and Mathematics). Still, they also complement part of the curriculum of the aforementioned social subjects. The workshops introduce the student to the study of Astronomy through activities that explain the apparent movement of the stars, the movement of the Sun, prominent constellations, the concept of the Zodiac, management of the planisphere, movement of the Moon, equation of time and construction of solar clocks.

In the study of constellations, we often use maps that identify their shapes and locate them to other constellations. In our workshops, we provide the students with blank maps that describe parts of the sky, and they identify the stars that conform to each constellation. These maps, known as planispheres, are usually the result of a stereographic or orthogonal projection of a fixed-radius sphere

in which the stars and other objects in the sky are arranged. This sphere is a classical model for representing the stars that have been used since ancient times by Hipparchus, Aristarchus, Ptolemy and others (Martín, 1990), and it is known in positional astronomy as the celestial sphere (Karttunen et al., 2007). However, we have noticed that the single use of these maps is not enough in learning the position of the stars as objects of the celestial sphere. A 3D model could significantly improve the spatial vision of the student and complement the learning we carry out with flat maps of many other features of the sky such as the size (*angular area*) of each constellation, opposite constellations, and the elliptic curve (*track of the apparent position of the Sun in the sky over a year*).

With this objective, the article summarizes the modeling of a celestial sphere in GeoGebra (Chan, 2013; Hohenwarter, 2004) that we have developed to work on these contents with high school students. The celestial sphere model is an ideal representation of the stars that considers them all to be confined at a fixed distance from the Earth and allows the basic modeling of the apparent daily and annual motion of the stars, planets, and the Sun. The celestial sphere takes a point close to the Polaris star as a north pole (North Celestial Pole) and its antipodal point as a south pole (South Celestial Pole). It considers an imaginary axis that passes through these two poles, which is an extension of the Earth's rotation axis, and assumes that the sphere presents a clockwise rotation movement about this axis opposite to the direction of the Earth. This assumption paves the way to model the apparent motion of the stars. The planet Earth is considered at the center of the sphere, although, due to astronomical distances, the Sun can also be considered the center of the sphere. This is the convention that we adopt in the article for the later purpose of understanding the concept of the Zodiac.

The model we present has been developed in GeoGebra and has the following characteristics:

- incorporates the representation of the 88 constellations of the modern almagest;
- incorporates the position of the ecliptic;
- considers the Sun at the center of the sphere and positions the planet Earth accordingly to model the apparent position of the Sun;
- incorporates the rotation motion of the celestial sphere and the orbital motion of the Earth and the apparent position of the Sun in the ecliptic; and
- incorporates the identification of the zodiac constellations.

However, it assumes the next simplifications that we consider necessary for the student to whom the dynamic construction is directed.

- Models the orbital motion of the Earth as a uniform circular motion; thus the orbit of the Earth is a circle nor an ellipse as stated by the traditional gravitation theory.
- The model does not incorporate the nutation motion nor the precession motion of the Earth, and thus it does not represent the precession of the equinoxes.

The following few sections describe our model, with each section focusing on a different issue. The first section describes the mathematics we use to model the various objects of the sphere. The second section describes how to build the sphere and the programming of constellations using the GeoGebra API in JavaScript. In the third section, we explain how to introduce the functionality in the application

by assigning actions to the already created buttons. As can be seen, the former sections are oriented to those teachers, and GeoGebra users who would like to make a similar application. The last section explains the application from a didactic point of view, emphasizing aspects that might be outlined when using the application with students. For example, the form we see in the celestial sphere is the mirror image of the one that an observer can see in the night sky. We also suggest other didactic uses of the application: (i) the identification of the position of the Sun in the sphere over a year and (ii) the identification of the Zodiac constellations. We finally conclude our work and suggest future development ideas.

2 SPHERE MODELING

2.1 Sphere, poles and meridians

We model the celestial sphere as a sphere with radius r centered at the origin $(0; 0; 0)$, by $x^2+y^2+z^2 = r^2$ and identify the celestial North and South poles as vectors $\mathbf{p}_n = (0; 0; r)^T$ and $\mathbf{p}_s = (0; 0; \quad r)^T$, respectively. In the celestial equator (*intersection of the celestial sphere and the equatorial plane*) we identify 24 hour points $\mathbf{p}_k$ that correspond to the hours of the meridians (also as vectors)

$$\mathbf{p}_k = (r\cos(15k); r\sin(15k); 0)^T \quad \text{for} \quad 0 \quad k \quad 23:$$

With these points we model the celestial meridians with the GeoGebra command `circle` through three given points.

$$\text{Circle}(\mathbf{p}_n; \mathbf{p}_s; \mathbf{p}_k) \quad \text{for} \quad 0 \quad k \quad 23:$$

We simulate the motion of the celestial sphere considering a matrix that describes a rotation of angle a ($0 \quad a \quad 360$) about the z-axis and applying this transformation to the points $\mathbf{p}_k$. We declare this parameter a as a slider and redefine the hour points this way, obtaining a meridian circle that incorporates this transformation. The result is the rotation of the celestial sphere when the slider is animated. We must take into account that the angle is measured in positive sense, and thus the motion of the celestial sphere may be done in contrary sense, a feature that can be done in the properties of the object.

The matrix that describes a rotation $T : \mathbb{R}^3 \; ! \; \mathbb{R}^3$ about the z-axis of angle a has three columns corresponding to the images of the standard basis vectors, $\mathbf{e}_1 = (1; 0; 0)^T; \mathbf{e}_2 = (0; 1; 0)^T$, and $\mathbf{e}_3 = (0; 0; 1)^T$.

$$T\mathbf{e}_1 = \begin{pmatrix} \cos(a) \\ \sin(a) \\ 0 \end{pmatrix}; \qquad T\mathbf{e}_2 = \begin{pmatrix} \sin(a) \\ \cos(a) \\ 0 \end{pmatrix}; \qquad \text{and} \qquad T\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}:$$

Thus, the matrix A of the linear transformation T that describes a rotation of angle a around the vertical axis is

$$A = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The point $\mathbf{p}_k$ rotates about the z-axis at the given a angle. The coordinates of this point are

$$Ap_k = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r\cos(15k) \\ r\sin(15k) \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos(a)\cos(15k) - r\sin(a)\sin(15k) \\ r\sin(a)\cos(15k) + r\cos(a)\sin(15k) \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos(15k + a) \\ r\sin(15k + a) \\ 0 \end{pmatrix}$$

2.2 Stars and constellations

Each star is identified by its *right ascension* (*angular separation of a star from the vernal equinox, a specific point of the celestial equator*) and *declination* (*angular separation of a star from the equatorial plane* (Karttunen et al., 2007)), which determine the spherical coordinates of the star. With this information we create a star-point object whose rectangular coordinates are

$$s = \begin{pmatrix} r\cos(\ )\cos(\ ) \\ r\cos(\ )\sin(\ ) \\ r\sin(\ ) \end{pmatrix}$$

Like the hour points, the star points must rotate on the sphere. Thus, the coordinates that must be considered in the modeling are the result of applying the linear transformation T to the previous vector, obtaining

$$As = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r\cos(\ )\cos(\ ) \\ r\cos(\ )\sin(\ ) \\ r\sin(\ ) \end{pmatrix} = \begin{pmatrix} r\cos(a)\cos(\ )\cos(\ ) - r\sin(a)\cos(\ )\sin(\ ) \\ r\sin(a)\cos(\ )\cos(\ ) + r\cos(a)\cos(\ )\sin(\ ) \\ r\sin(\ ) \end{pmatrix}$$

After creating the stars, we build each constellation as polygons through its stars.

2.3 Ecliptic, Sun and Earth

We model the ecliptic (*apparent orbit of the Sun in the sky*) like a circle. To do this, we identify three of its points and build the circle they determine. For two of these points, we consider the Aries point and the Libra point, coinciding with the spring and autumn equinoxes and defined as the intersection of the ecliptic with the celestial equator. Given the modeling we have chosen for our development, the Aries point corresponds to the point p_0 and the Libra point to the point p_{12} in the celestial equator. For the initial instant of the animation of the rotation motion ($a = 0$), these points correspond to $(r; 0; 0)^T$ and $(r; 0; 0)^T$. We choose the summer solstice in the northern hemisphere for the third point of the construction. This point is defined as the highest one in the ecliptic, and its coordinates, at the initial time of the animation, are those of the image of point $(0; r; 0)^T$ under a rotation of (*obliquity of the ecliptic*) around the x-axis.

Arguing as in previous paragraphs, we get the fundamental matrix B that defines this rotation. In addition, we notice that the vernal and autumn equinoxes are fixed points for this transformation.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\ ) & \sin(\ ) \\ 0 & \sin(\ ) & \cos(\ ) \end{pmatrix}$$

Thus, the coordinates of the three points we need for the construction of the ecliptic are those of the application of the composition of the rotation about the x-axis (B) and the rotation about the vertical axis (A) to points $(r; 0; 0)^T$, $(0; r; 0)^T$ and $(\ \ r; 0; 0)^T$. We get three points in the ecliptic, which move accordingly to the rest of the stars.

$$E_1 = AB \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\ ) & \sin(\ ) \\ 0 & \sin(\ ) & \cos(\ ) \end{pmatrix} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos(a) \\ r\sin(a) \\ 0 \end{pmatrix}$$

$$E_2 = AB \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\ ) & \sin(\ ) \\ 0 & \sin(\ ) & \cos(\ ) \end{pmatrix} \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ r\cos(\ ) \\ r\sin(\ ) \end{pmatrix} = \begin{pmatrix} r\sin(a)\cos(\ ) \\ r\cos(a)\cos(\ ) \\ r\sin(\ ) \end{pmatrix}$$

$$E_3 = AB \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\ ) & \sin(\ ) \\ 0 & \sin(\ ) & \cos(\ ) \end{pmatrix} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos(a) \\ r\sin(a) \\ 0 \end{pmatrix}$$

Once we have identified these points, we build the ecliptic as the circle they determine.

$$Ecliptic = Circle(E_1; E_2; E_3)$$

To model the apparent Sun (*position of the Sun in the sphere as seen from Earth*), we consider a point in the celestial equator depending on a parameter t ($0 \quad t \quad 360$) and locate it in the ecliptic applying the rotation B. Then, we rotate it by using the rotation A. This parameter t is modeled with positive velocity to maintain the coherence of the annual motion of the Sun on the celestial sphere. We thus obtain the following.

$$\text{Sun}_A = AB \begin{pmatrix} r\cos(t) \\ r\sin(t) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\ ) & \sin(\ ) \\ 0 & \sin(\ ) & \cos(\ ) \end{pmatrix} \begin{pmatrix} r\cos(t) \\ r\sin(t) \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r\cos(t) \\ r\cos(\ )\sin(t) \\ r\sin(\ )\sin(t) \end{pmatrix} = \begin{pmatrix} r\cos(a)\cos(t) & r\sin(a)\cos(\ )\sin(t) \\ r\sin(a)\cos(t) + r\cos(a)\cos(\ )\sin(t) \\ r\sin(\ )\sin(t) \end{pmatrix}$$

In a similar way we model the orbital motion of Earth, shifting the slider 180 and downsizing the radius r to r' for the simulation. We thus obtain.

$$\text{Earth} = \begin{pmatrix} r'\cos(a)\cos(t + 180\ ) & r'\sin(a)\cos(\ )\sin(t + 180\ ) \\ r'\sin(a)\cos(t + 180\ ) + r'\cos(a)\cos(\ )\sin(t + 180\ ) \\ r'\sin(\ )\sin(t + 180\ ) \end{pmatrix}$$