# A COLUMN-VECTOR GEOMETRIC INTERPRETATION OF LINEAR SYSTEMS AND GAUSSIAN ELIMINATION

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**Abstract**

*This paper presents GeoGebra constructions that provide a visual and interactive interpretation of linear systems and Gaussian elimination from the vector-and-matrix viewpoint of linear algebra. The approach emphasizes the column-vector interpretation of* $\mathsf{Ax} = \mathsf{b}$*: solving a linear system means expressing the right-hand side as a linear combination of the columns of the coefficient matrix. In this setting, a fundamental row operation in Gaussian elimination has a natural geometric counterpart: a shear transformation. This perspective makes visible the geometric meaning of row reduction and clarifies its connection with the* $\mathsf{LU}$ *factorization of matrices. The GeoGebra applets support dynamic exploration of the links among algebraic procedures, vector combinations, matrix operations, and geometric transformations.*

**Keywords**: linear systems; GeoGebra; shear transformations; Gaussian elimination; LU factorization.

## 1 Introduction

Linear systems are a central topic in school mathematics, college algebra, and linear algebra, with applications in science, engineering, economics, computer graphics, data science, and many other fields. In classrooms, the topic is often introduced through algebraic procedures: substitution, elimination, graphing, row reduction, and matrix methods. These procedures are important, but a geometric perspective can provide a complementary way to understand what the computations are doing.

In this paper, we develop a geometric interpretation of Gaussian elimination in which elementary row operations are viewed as shear transformations acting on the column vectors of a matrix and on the right-hand side vector. Instead of interpreting a $2 \times 2$ system only by associating one line with each equation, we also read the coefficients vertically and interpret the columns of the coefficient matrix as vectors. In this column-vector interpretation, solving a system means finding the coefficients of a linear combination that produces the right-hand side vector. Interactive GeoGebra applets support this transition between algebraic procedures and geometric visualization, allowing students and teachers to move between symbolic expressions, matrices, vectors, coordinate systems, and dynamic transformations.

In the United States, systems of linear equations appear prominently in the transition from middle school algebra to high school algebra. The Common Core State Standards for Mathematics (CCSSM) identify, for Grade 8, the analysis and solution of pairs of simultaneous linear equations, including the interpretation of solutions as intersection points of graphs and the use of algebraic and graphical methods Common Core State Standards Initiative (2010). At the high school level, the standards include solving systems of equations, justifying the elementary replacement of one equation by the sum of that equation and a multiple of another, representing a system of linear equations as a matrix equation in a vector variable, and using matrices to solve linear systems when appropriate Common Core State Standards Initiative (2010). Thus the interpretation developed here connects naturally with both the usual graphical approach to systems and the later vector-and-matrix approach of linear algebra.

This adaptation also takes into account a distinctive feature of the Common Core framework: the standards specify mathematical content and practices, but they do not prescribe a single curriculum sequence or a single pedagogy. The high school standards may be organized in a traditional sequence such as Algebra I, Geometry, and Algebra II, or in an integrated sequence such as Mathematics 1, Mathematics 2, and Mathematics 3 Common Core State Standards Initiative (2010). The approach proposed in this paper can be used in either setting: as enrichment for systems of equations in Algebra I or Integrated Mathematics, as preparation for matrix methods in Algebra II or Precalculus, or as a visualization of Gaussian elimination in a first course in linear algebra.

The Standards for Mathematical Practice provide an additional motivation for the use of dynamic geometry software. The present approach asks students to reason abstractly and quantitatively, use appropriate tools strategically, and look for structure in algebraic procedures Common Core State Standards Initiative (2010). In this sense, the goal is not to replace algebraic fluency, but to make the underlying structure of Gaussian elimination visible.

The usual geometric interpretation of a $2 \times 2$ linear system associates one line with each equation. The relative positions of the two lines in the coordinate plane determine the nature of the solution set. For example, consider the system

$$\begin{cases} 2s - t = 1, \\ s + t = 5, \end{cases} \qquad s, t \in \mathbb{R}. \tag{1}$$

The first equation corresponds to the blue line in Figure 1, and the second equation corresponds to the red line. Since the lines intersect, the coordinates of the point of intersection $P = (s, t) = (2, 3)$ give the unique solution of the system.

## 2 An Alternative Geometric Interpretation

There is another, less common way to interpret the same linear system geometrically. Observe that

$$\begin{cases} 2s - t = 1, \\ s + t = 5 \end{cases} \iff \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \iff s \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Thus solving the system can be interpreted as a linear-combination problem: determine whether the vector $\mathbf{w} = (1, 5)$ can be written as a linear combination of the vectors $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (-1, 1)$. In this example, the answer is yes, and the coefficients of the linear combination are exactly $s = 2$ and $t = 3$, the solution of the original system. The reader can explore the two interpretations in the GeoGebra applet shown in Figure 1.

Strang (2008) emphasizes that it is essential to see $Ax$ as a combination of the columns of $A$. This leads directly to the idea of the column space and provides a powerful geometric interpretation of linear systems.

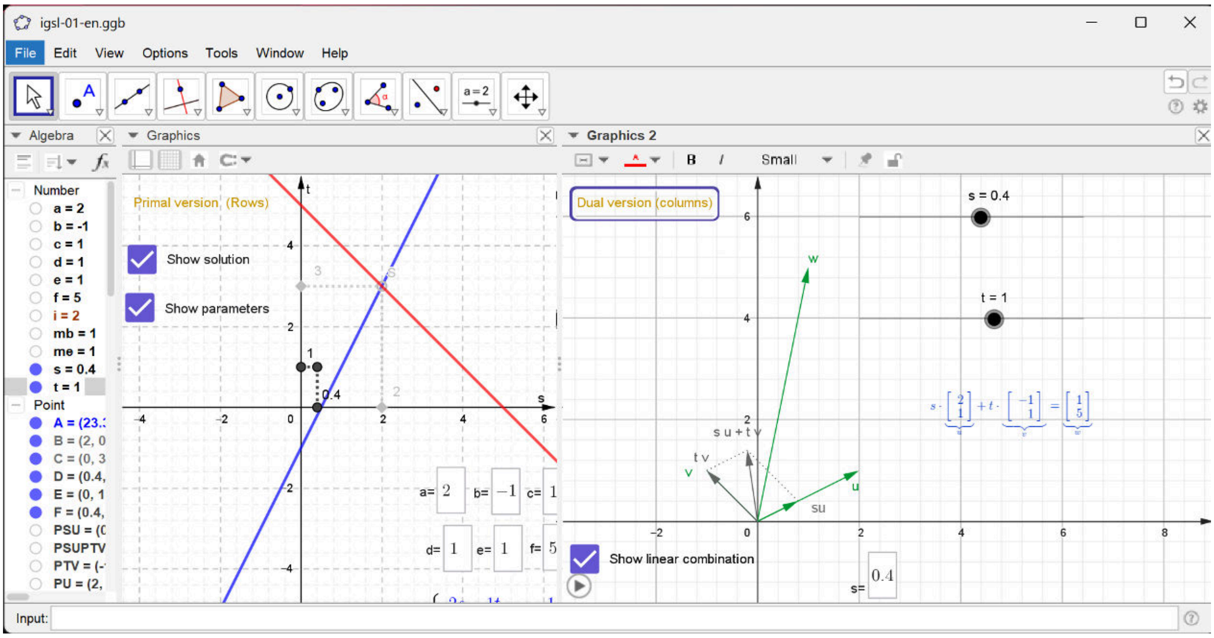



**Figure 1.** Interactive applet for exploring two geometric interpretations of a $2 \times 2$ linear system: `https://www.im-uff.mat.br/nagj/igslvv/igsl-01-en.ggb`.

**Source:** The authors.

The column-vector viewpoint introduced above also sets the stage for a geometric interpretation of the elementary row operations used in Gaussian elimination. In the next section, we show that these operations can be viewed as shear transformations applied simultaneously to the column vectors of the coefficient matrix and to the right-hand side vector.

## 3 Shears and Gaussian Elimination

Before giving the formal definition, it is helpful to think of a shear as a controlled sliding motion. In a horizontal shear, points slide parallel to the $x$-axis, and the amount of sliding depends on their $y$-coordinate. In a vertical shear, points slide parallel to the $y$-axis, and the amount of sliding depends on their $x$-coordinate. As we shall see, the row-replacement operations used in Gaussian elimination can be interpreted geometrically as such shearing motions applied to the column vectors of a matrix.

**Definition 1.** *A* horizontal shear *of ratio* $p$ *in the plane, that is, a shear in the direction of the* $x$*-axis, is a linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *of the form*

$$(\tilde{x}, \tilde{y}) = T(x, y) = (x + py, y).$$

*In matrix form,*

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

*A* vertical shear *of ratio* $p$ *in the plane, that is, a shear in the direction of the* $y$*-axis, is a linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *of the form*

$$(\tilde{x}, \tilde{y}) = T(x, y) = (x, px + y).$$

*In matrix form,*

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Horizontal shears transform axis-aligned squares into parallelograms with the same base and height, and therefore preserve area. Vertical shears similarly transform axis-aligned squares into parallelograms and preserve area. These properties can be visualized in the GeoGebra applet shown in Figure 2.[1]

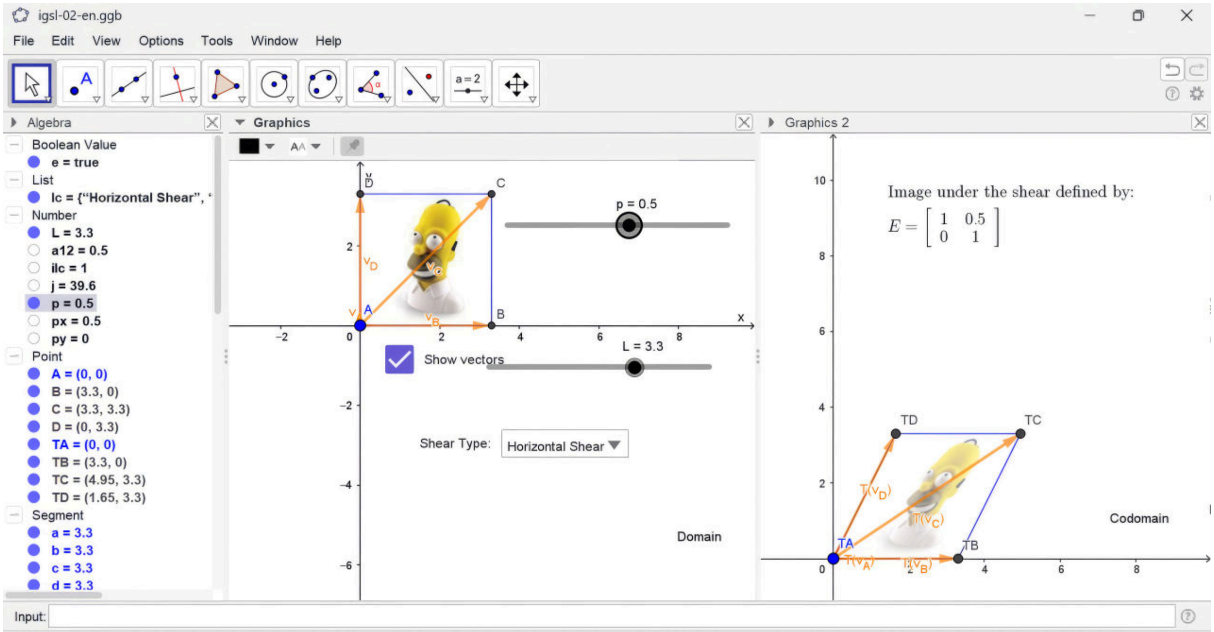



**Figure 2.** Interactive applet for exploring shears in the plane: `https://www.im-uff.mat.br/nagj/igslvv/igsl-02-en.ggb`.

**Source:** The authors.

**Proposition 2.** *(a) A vertical shear of ratio* $p$ *in the plane is an invertible linear transformation, and its inverse is a vertical shear of ratio* $-p$.

*(b) A horizontal shear of ratio* $p$ *in the plane is an invertible linear transformation, and its inverse is a horizontal shear of ratio* $-p$.

*Proof.* In the standard basis, the matrix associated with a vertical shear of ratio $p$ is

$$A = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix}.$$

This matrix is invertible, and its inverse is

$$B = \begin{bmatrix} 1 & 0 \\ -p & 1 \end{bmatrix}.$$

[1] Shear transformations also appear in several other mathematical and applied contexts. They can be used in area-based proofs of the Pythagorean theorem; see, for example, `https://www.geogebra.org/m/xCYQdz5g`. In computer graphics, Paeth's rotation algorithm decomposes a rotation into a sequence of three shears Schneider and Eberly (2002). Shears also occur in typography, when upright fonts are transformed into oblique fonts, and in classical mechanics, where Galilean transformations may be viewed as shears Halliday et al. (2016).

The matrix $B$ is precisely the matrix associated with a vertical shear of ratio $-p$. This proves part (a). The proof of part (b) is analogous. $\square$

For linear systems with more than two variables, we need shears that act in coordinate planes. For example, in $\mathbb{R}^3$ we use matrices such as

$$E_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{xz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p & 0 & 1 \end{bmatrix}, \qquad E_{yz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p & 1 \end{bmatrix}.$$

The matrix $E_{xy}$ defines a shear in the direction of the $y$-axis in the $xy$-plane, leaving the $z$-component unchanged. The matrix $E_{xz}$ defines a shear in the direction of the $z$-axis in the $xz$-plane, leaving the $y$-component unchanged. Finally, $E_{yz}$ defines a shear in the direction of the $z$-axis in the $yz$-plane, leaving the $x$-component unchanged. These matrices are invertible, and their inverses are shears of the same type:

$$E_{xy}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -p & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{xz}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p & 0 & 1 \end{bmatrix}, \qquad E_{yz}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -p & 1 \end{bmatrix}.$$

The classical method of Gaussian elimination solves a linear system by applying elementary row operations to its augmented matrix until a row echelon form is obtained. For the system (1), we have

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & 5 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & \frac{3}{2} & \frac{9}{2} \end{array}\right].$$

The elementary operation $R_2 \leftarrow R_2 - \frac{1}{2}R_1$ can be encoded as left multiplication by a matrix. Let

$$E = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} 2 & -1 \\ 0 & \frac{3}{2} \end{bmatrix}.$$

Then

$$EA = U.$$

The matrix $E$ is lower triangular and defines a vertical shear. The matrix $U$ is upper triangular. Since $E$ is invertible, its inverse $L = E^{-1}$ is also a shear:

$$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

Consequently,

$$EA = U \quad \Longrightarrow \quad LEA = LU \quad \Longrightarrow \quad IA = LU \quad \Longrightarrow \quad A = LU.$$

Gaussian elimination therefore expresses $A$ as the product of a lower triangular matrix $L$, with all diagonal entries equal to 1, and an upper triangular matrix $U$. Geometrically, left multiplication by $E$ applies the same shear to the column vectors of $A$ and to the right-hand side vector. In particular, it maps $\mathbf{u} = (2, 1)$ to $(2, 0)$, a vector parallel to the $x$-axis. After this shear, the corresponding linear system is simpler to solve because one of the associated vectors has been aligned with a coordinate axis. Figure 3 illustrates this geometric effect: the shear transformation moves the column vectors and the right-hand side vector in a coordinated way, while preserving the coefficients of the linear

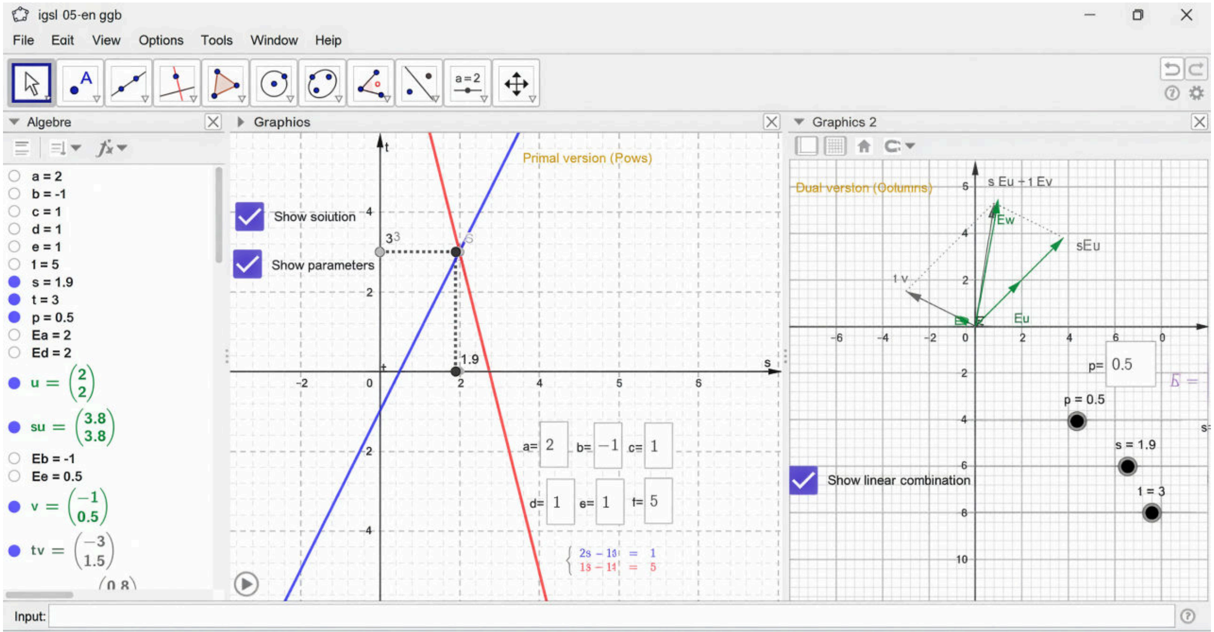



**Figure 3.** Interactive applet for exploring how the shear transformation associated with a fundamental step of Gaussian elimination aligns one column vector with a coordinate axis, thereby simplifying the corresponding linear system: `https://www.im-uff.mat.br/nagj/igslvv/igsl-05-en.ggb`.

**Source:** The authors.

combination. In particular, when the appropriate shear ratio is chosen, the vector $\mathbf{u} = (2, 1)$ is aligned with the $x$-axis, simplifying the corresponding linear system. The reader can explore this transformation dynamically in the applet by using the slider $p$ to adjust the shear ratio.

Let us consider a second example, now in three dimensions. Consider the linear system

$$\begin{cases} 4x - 2y + 2z = 5, \\ 2x + 4y = 5, \\ x + 2y + 4z = 5. \end{cases}$$

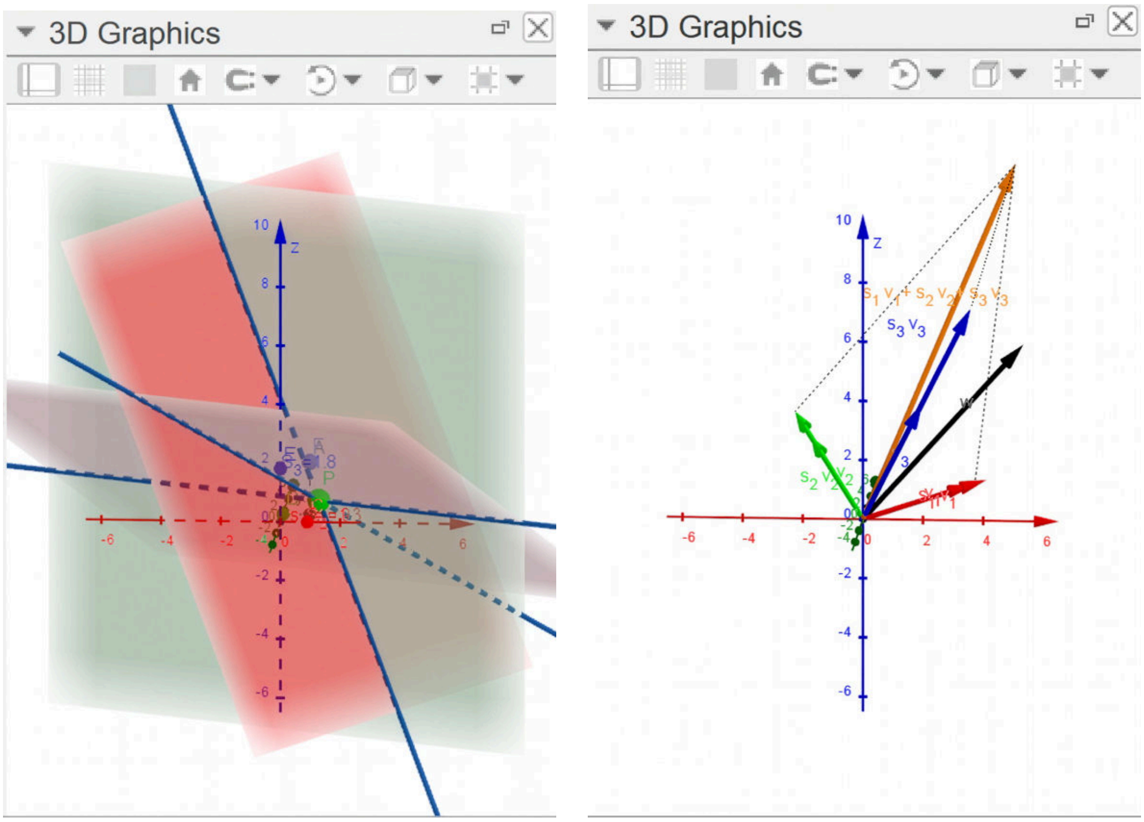
In the row interpretation, each equation represents a plane, and solving the system means finding the point of intersection of the three planes. In the column interpretation, define

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}.$$

Solving the system means finding scalars $x$, $y$, and $z$ such that

$$\mathbf{w} = x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3.$$

Figure 4 compares the two geometric viewpoints in this three-dimensional example. The row interpretation represents the equations as planes, while the column interpretation represents the system as a linear-combination problem involving vectors. This comparison helps explain why the column-vector viewpoint can provide a less visually cluttered setting for discussing Gaussian elimination geometrically.




Row interpretation | Column interpretation

**Figure 4.** Comparison between the row interpretation and the column-vector interpretation for a system of three equations in three variables: `https://www.im-uff.mat.br/nagj/igslvv/igsl-06-en.ggb`.

**Source:** The authors.

Now row-reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 4 & -2 & 2 & 5 \\ 2 & 4 & 0 & 5 \\ 1 & 2 & 4 & 5 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 4 & -2 & 2 & 5 \\ 0 & 5 & -1 & \frac{5}{2} \\ 1 & 2 & 4 & 5 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - \frac{1}{4}R_1} \left[\begin{array}{ccc|c} 4 & -2 & 2 & 5 \\ 0 & 5 & -1 & \frac{5}{2} \\ 0 & \frac{5}{2} & \frac{7}{2} & \frac{15}{4} \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 4 & -2 & 2 & 5 \\ 0 & 5 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & \frac{5}{2} \end{array}\right].$$

Geometrically, each of these three row operations is a shear in a coordinate plane of the type introduced earlier: the operation $R_2 \leftarrow R_2 - \frac{1}{2}R_1$ is a shear of type $E_{xy}$, acting in the $xy$-plane; $R_3 \leftarrow R_3 - \frac{1}{4}R_1$ is a shear of type $E_{xz}$, acting in the $xz$-plane; and $R_3 \leftarrow R_3 - \frac{1}{2}R_2$ is a shear of type $E_{yz}$, acting in the $yz$-plane. The reduction to upper triangular form is therefore the result of applying these three shears successively to the column vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and to $\mathbf{w}$.

As in the $2 \times 2$ case, the elementary operations above can be encoded by matrices:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}}_{E_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 4 & -2 & 2 \\ 2 & 4 & 0 \\ 1 & 2 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 4 & -2 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 4 \end{bmatrix}}_{U}.$$

The matrices $E_1$, $E_2$, and $E_3$ are invertible lower triangular shear matrices with diagonal entries equal to 1. Their inverses are

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{bmatrix}, \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}.$$

We now examine how these elementary shear matrices combine to produce the $LU$ factorization. Since $E_3E_2E_1A = U$, it follows that $A = LU$, where

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}.$$

It is important that the elementary operations be carried out in the correct order to obtain the desired structure of $L$: lower triangular with all diagonal entries equal to 1. More precisely, the operations are performed in the order

$$R_2 \leftarrow R_2 - l_{21}R_1, \qquad R_3 \leftarrow R_3 - l_{31}R_1, \qquad \text{and} \qquad R_3 \leftarrow R_3 - l_{32}R_2.$$

The matrices that encode these operations are, respectively,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{31} & 0 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l_{32} & 1 \end{bmatrix}.$$

Thus $E_3E_2E_1A = U$ is upper triangular, and

$$\begin{aligned}
A &= E_1^{-1}E_2^{-1}E_3^{-1}U \\
&= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix} U \\
&= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} U \\
&= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L} U.
\end{aligned}$$

The computations above also show where the entries below the main diagonal of $L$ come from: they are precisely the coefficients $l_{21}$, $l_{31}$, and $l_{32}$ used in the elementary row operations of Gaussian elimination.

From this viewpoint, the $LU$ factorization emerges naturally from the geometry of shear transformations. The matrix $U$ is obtained by successively applying elementary shears that simplify the column-vector configuration, while the lower triangular factor $L$ is the composition of the inverse shear transformations that recover the original matrix.

## 4 Final Remarks

The column-vector interpretation of a linear system establishes a geometric context based on vectors and makes visible the geometric counterpart of Gaussian elimination: shear transformations. In this viewpoint, elementary row-replacement operations are not only algebraic steps performed on an augmented matrix; they can also be understood as geometric transformations applied simultaneously to the column vectors of the coefficient matrix and to the right-hand side vector.

From this perspective, the $LU$ factorization acquires a natural geometric meaning. The upper triangular matrix $U$ is obtained by applying elementary shear transformations that simplify the column-vector configuration, while the lower triangular factor $L$ records the inverse shears that recover the original matrix.

From an instructional point of view, the approach developed here can support several goals that are important in classrooms guided by the Common Core framework: connecting graphical and algebraic representations of systems, highlighting the invariance of the solution set under elementary equation operations, preparing students for matrix representations of systems, and using dynamic technology to make mathematical structure visible. It is therefore well aligned with the Common Core emphasis on systems of equations, modeling, strategic use of tools, and the search for mathematical structure.

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