GEOGEBRA VISUALIZATIONS OF EIGENVALUES, EIGENVECTORS, AND EIGENSPACES OF SYMMETRIC MATRICES

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Abstract

This article presents four GeoGebra-based visualizations of symmetric matrices designed to support the teaching and learning of eigenvalues, eigenvectors, and eigenspaces. The examples include two-dimensional and three-dimensional matrices, with both invertible and singular cases, as well as distinct and repeated eigenvalues. Students compare eigenvectors with their images under the matrix transformation and observe stretching, direction reversal, direction preservation, and collapse to the zero vector. The constructions also illustrate the orthogonality of eigenspaces associated with distinct eigenvalues, the connection between a zero eigenvalue and the nullspace, and the geometric meaning of a repeated eigenvalue. In the repeated-eigenvalue example, GeoGebra shows that an entire two-dimensional zero-eigenspace is mapped to the zero vector, rather than only the individual eigenvectors displayed. Classroom prompts and extension activities encourage students to move beyond symbolic calculation and coordinate algebraic, geometric, and visual representations. The visualizations may help students develop a more conceptual understanding of symmetric matrices and their eigenspaces.

Keywords: Symmetric matrices, eigenvalues, eigenvectors, eigenspaces, GeoGebra, linear algebra education, dynamic visualization, repeated eigenvalues

1 Introduction

Symmetric matrices play a fundamental role in linear algebra, particularly in the study of eigenvalues and eigenvectors. A matrix A is called symmetric when $A^T = A$; in the complex case, the analogous condition is self-adjointness, $A^* = A$, where A^* denotes the conjugate transpose of A. Symmetric matrices have several important properties that are central to introductory and intermediate linear algebra courses: their eigenvalues are real, eigenvectors corresponding to distinct eigenvalues are orthogonal, and the resulting eigenspaces provide a geometrically meaningful way to understand the action of the matrix.

The visualization of symmetric matrices, particularly their eigenvalues, eigenvectors, and eigenspaces, offers an intuitive way to grasp the underlying geometric structures. Traditional symbolic methods often emphasize computing the characteristic polynomial and solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$, but students may not immediately connect these symbolic procedures to the geometric meaning of the equation $A\mathbf{v} = \lambda\mathbf{v}$. In that equation, the vector $\mathbf{v}$ is not sent to an arbitrary new direction; rather, it remains on the same line through the origin and is stretched, compressed, reversed, or collapsed according to the eigenvalue λ.

This emphasis on coordinating symbolic, geometric, and visual representations is consistent with work in linear algebra education that highlights students' need to move flexibly among algebraic, geometric, and abstract modes of description (Hillel, 2000; Dorier & Sierpinska, 2001). It is also aligned with Harel's (2000) principles for the teaching and learning of linear algebra, particularly the principles of concreteness, necessity, and generalizability: students benefit from working with models that make mathematical structures experientially meaningful, from encountering situations that create a need for the relevant ideas, and from using those models as a basis for generalization. In the present GeoGebra constructions, these principles are reflected in opportunities for students to compare an input vector with its image under a matrix transformation, observe direction preservation or collapse to the zero vector, and use specific visual cases as a basis for broader conclusions about eigenspaces and symmetric matrices.

This note presents four GeoGebra-based visualizations of symmetric matrices: two in two dimensions and two in three dimensions. The first case considers an invertible 2×2 symmetric matrix with distinct eigenvalues. The second case considers a singular 2×2 symmetric matrix and highlights the geometric meaning of a zero eigenvalue. The third case extends the construction to an invertible 3×3 symmetric matrix with three distinct eigenvalues. The final case considers a singular 3×3 symmetric matrix with a repeated eigenvalue, emphasizing eigenspaces, basis choice, and the distinction between algebraic and geometric multiplicity. Across the four examples, GeoGebra is used not only to confirm known theoretical results, but also to support inquiry into how eigendirections, nullspaces, and invariant subspaces can be seen dynamically.

2 Pedagogical design of the GeoGebra constructions

The constructions in this note are designed for use in a linear algebra classroom after students have been introduced to eigenvalues and eigenvectors symbolically. Each construction follows the same pedagogical sequence. First, a symmetric matrix A is defined in GeoGebra. Second, candidate eigenvectors $\mathbf{v}_i$ are entered as vectors. Third, the transformed vectors $A\mathbf{v}_i$ are constructed. Finally, students compare $\mathbf{v}_i$ and $A\mathbf{v}_i$ by observing their directions, measuring their angles, and interpreting the scale factor relating them.

This sequence supports the following learning objectives. Students should be able to:

1. interpret the equation $A\mathbf{v} = \lambda\mathbf{v}$ geometrically as direction preservation, direction reversal, stretching, compression, or collapse to the zero vector;
2. explain why eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal;
3. distinguish between an individual eigenvector and the eigenspace to which it belongs;
4. interpret the zero eigenvalue in terms of the nullspace of the matrix; and
5. describe the repeated-eigenvalue case in terms of eigenspaces and basis choice rather than as a list of isolated vectors.

The repeated use of the same GeoGebra commands across the examples is intentional for classroom implementation; however, the mathematical discussion below emphasizes the differences among the four cases. The first example establishes the basic visual meaning of eigenvectors. The second shows how a singular matrix collapses a nonzero eigenvector to the zero

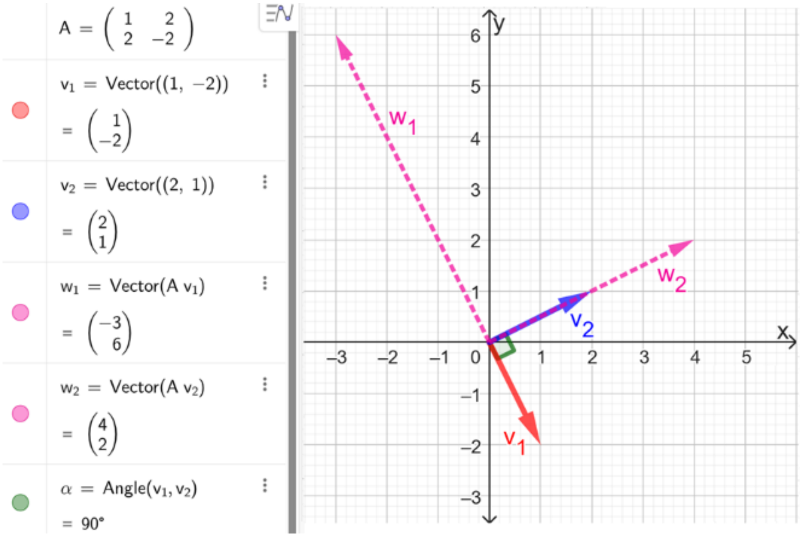

Figure 1. Visualizing a 2×2 invertible symmetric matrix.

vector. The third extends the same ideas to an orthogonal eigenbasis in $\mathbb{R}^3$. The fourth deepens the discussion by showing that a repeated eigenvalue may correspond to an entire plane of eigenvectors.

3 Visualizing a 2×2 invertible symmetric matrix with distinct eigenvalues

Consider the 2×2 symmetric matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}.$$

Its eigenvalues and eigenvectors can be found by solving the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0.$$

This gives the eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$. Substituting each eigenvalue into $(A - \lambda I)\mathbf{v} = \mathbf{0}$ gives, for example,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{for} \quad \lambda_1 = -3, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{for} \quad \lambda_2 = 2.$$

To visualize these eigenvectors in GeoGebra Classic, define the matrix using the command `A = {{1, 2}, {2, -2}}`. Then define the eigenvectors using `v_1 = Vector((1, -2))` and `v_2 = Vector((2, 1))`, which represent the red and blue vectors in Figure 1. To construct their images under the matrix transformation, run `w_1 = Vector(A v_1)` and `w_2 = Vector(A v_2)`. These commands produce the dashed vectors $A\mathbf{v}_1$ and $A\mathbf{v}_2$.

Figure 1 illustrates the basic geometric meaning of the eigenvector equation. The vector $A\mathbf{v}_1$ lies on the same line as $\mathbf{v}_1$ but points in the opposite direction because $A\mathbf{v}_1 = -3\mathbf{v}_1$. By contrast, $A\mathbf{v}_2$ lies on the same line as $\mathbf{v}_2$ and points in the same direction because $A\mathbf{v}_2 = 2\mathbf{v}_2$. Thus, the negative eigenvalue reverses direction and stretches the vector by a factor of 3, while the positive eigenvalue preserves direction and stretches the vector by a factor of 2.

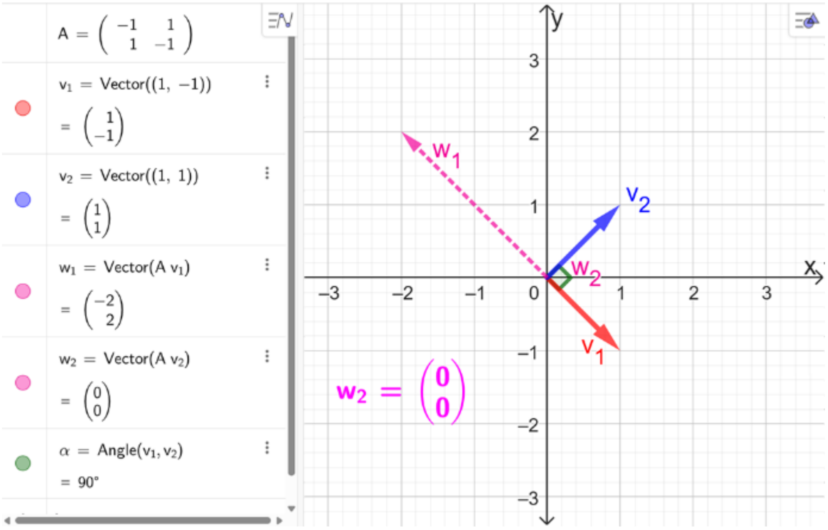

Figure 2. Visualizing a 2×2 singular symmetric matrix.

The figure also supports a visual confirmation of the symmetric matrix orthogonal eigenvectors theorem. By measuring the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, students can observe that the two eigendirections are perpendicular. A useful classroom prompt is: *Before measuring the angle, use the displayed matrix to predict whether the eigendirections should be orthogonal. Then use GeoGebra to test the prediction and explain how the visual evidence relates to the theorem.*

4 Visualizing a 2×2 singular symmetric matrix with distinct eigenvalues

The next example introduces a singular symmetric matrix:

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0,$$

which yields $\lambda_1 = -2$ and $\lambda_2 = 0$. One possible pair of corresponding eigenvectors is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{for} \quad \lambda_1 = -2, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for} \quad \lambda_2 = 0.$$

In GeoGebra Classic, enter `A = {{-1, 1}, {1, -1}}`, followed by `v_1 = Vector((1, -1))` and `v_2 = Vector((1, 1))`. Then construct the image vectors using `w_1 = Vector(A v_1)` and `w_2 = Vector(A v_2)`.

Figure 2 provides a useful contrast with the invertible case. For $\lambda_1 = -2$, the vector $A\mathbf{v}_1$ remains on the same line as $\mathbf{v}_1$ but points in the opposite direction, since $A\mathbf{v}_1 = -2\mathbf{v}_1$. For $\lambda_2 = 0$, however, the vector $\mathbf{v}_2$ is mapped to the zero vector: $A\mathbf{v}_2 = \mathbf{0}$. In this case, the transformed vector no longer has a visible direction, so the idea of parallelism is replaced by the idea of collapse to the origin.

This example helps students connect the zero eigenvalue to the nullspace. The line spanned by $\mathbf{v}_2$ is the eigenspace associated with $\lambda = 0$, and every vector on that line is sent to the zero vector. A useful exploration is to ask students to create additional scalar multiples of $\mathbf{v}_2$ and verify that each one collapses to the origin. This task addresses a common misconception: although the zero vector itself is not an eigenvector, a nonzero eigenvector can be mapped to the zero vector when its eigenvalue is zero.

5 Visualizing a 3×3 invertible symmetric matrix with distinct eigenvalues

We now extend the construction to three dimensions. Consider the 3×3 symmetric matrix

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 4 & 1 \\ 4 & 1 & 1 \end{bmatrix}.$$

Its characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 4 \\ 1 & 4-\lambda & 1 \\ 4 & 1 & 1-\lambda \end{vmatrix} = 0,$$

which gives the eigenvalues $\lambda_1 = -3$, $\lambda_2 = 3$, and $\lambda_3 = 6$. One corresponding choice of eigenvectors is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

In GeoGebra 3D, define the matrix with `A = {{1, 1, 4}, {1, 4, 1}, {4, 1, 1}}`. Then enter `v_1 = Vector((1, 0, -1))`, `v_2 = Vector((1, -2, 1))`, and `v_3 = Vector((1, 1, 1))`. Finally, construct their images using `w_1 = Vector(A v_1)`, `w_2 = Vector(A v_2)`, and `w_3 = Vector(A v_3)`.

Figure 3 shows how the two-dimensional observations generalize to an orthogonal eigenbasis in $\mathbb{R}^3$. Each displayed eigenvector is mapped to a scalar multiple of itself:

$$A\mathbf{v}_1 = -3\mathbf{v}_1, \qquad A\mathbf{v}_2 = 3\mathbf{v}_2, \qquad A\mathbf{v}_3 = 6\mathbf{v}_3.$$

Thus, the three eigendirections are invariant lines under the transformation. The transformation reverses and stretches the first eigendirection, while it stretches the second and third eigendirections without reversing them. Because the matrix is symmetric and the eigenvalues are distinct, the corresponding eigendirections are mutually orthogonal.

Figure 4 focuses on angle measurements among the three eigenvectors. Measuring the angles between $\mathbf{v}_1$ and $\mathbf{v}_2$, $\mathbf{v}_1$ and $\mathbf{v}_3$, and $\mathbf{v}_2$ and $\mathbf{v}_3$ gives $90°$ in each case. A useful student task is to rotate the 3D view and explain why the three eigendirections can be interpreted as a coordinate system adapted to the matrix transformation. This helps students see that the eigenvectors do not merely form three separate arrows; together they form an eigenbasis for $\mathbb{R}^3$.

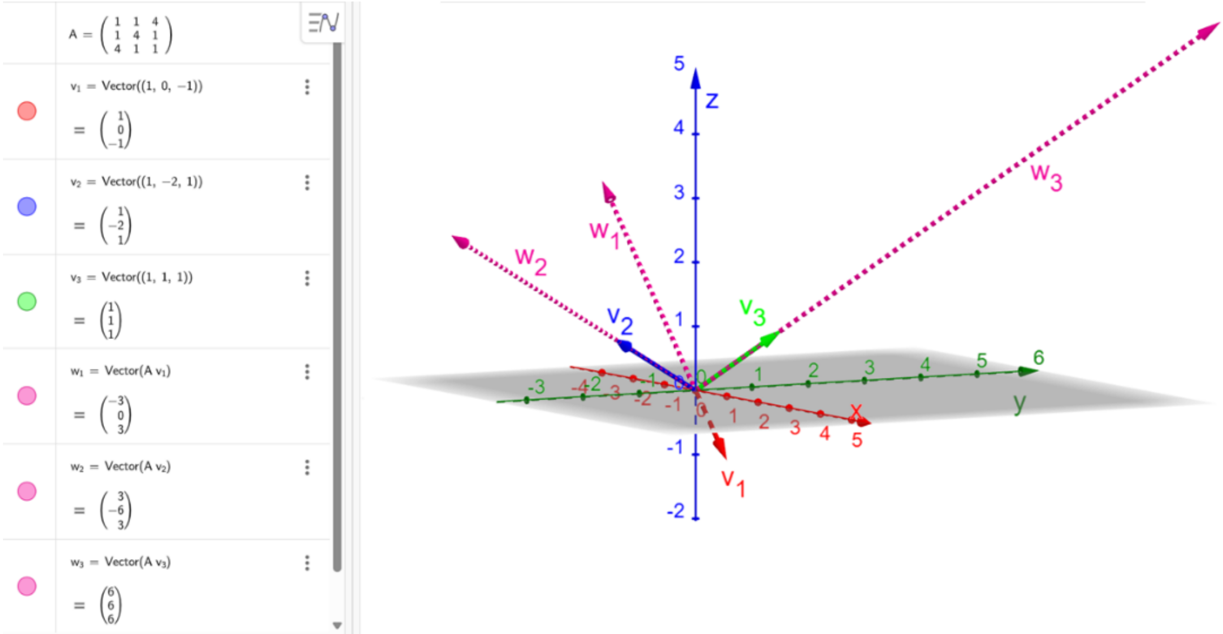

Figure 3. Visualizing a 3×3 invertible symmetric matrix with distinct eigenvalues.

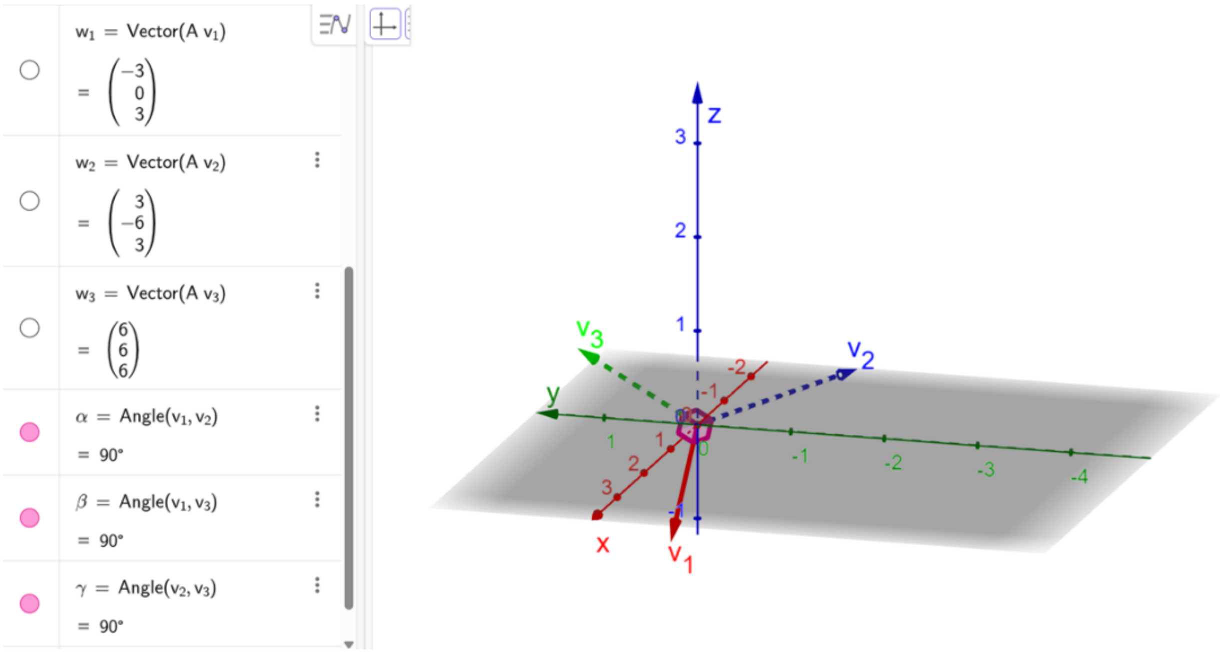

Figure 4. Visualizing the *symmetric matrix orthogonal eigenvectors* theorem.

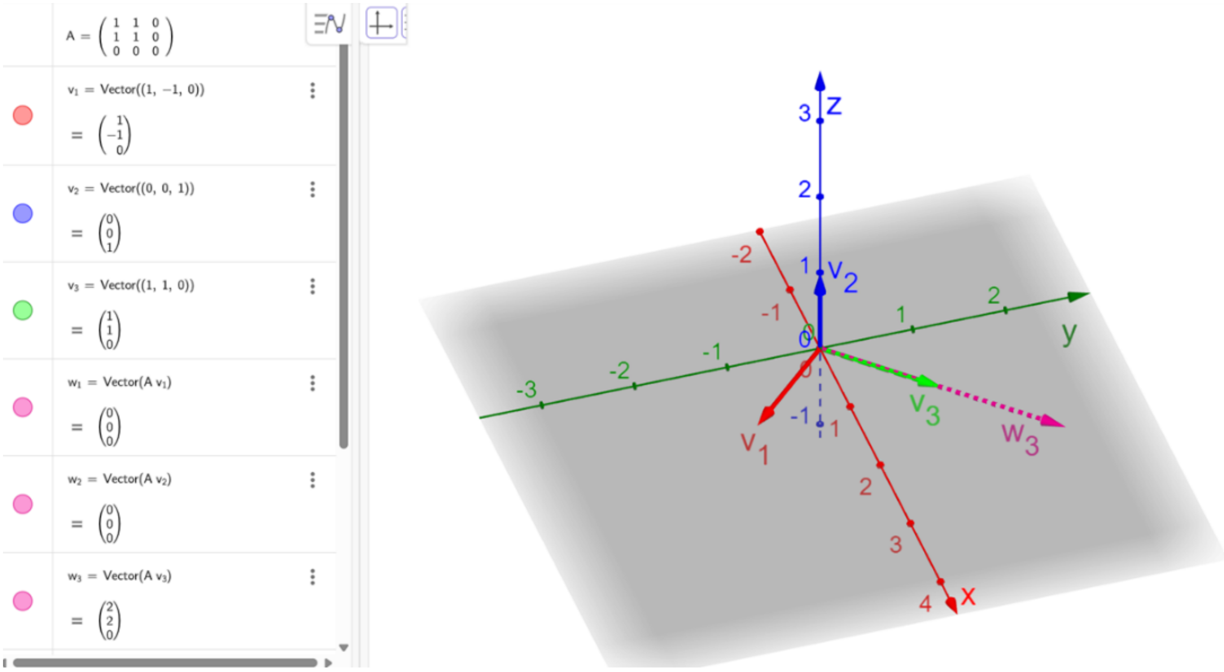

Figure 5. Visualizing a 3×3 singular symmetric matrix with repeated eigenvalues.

6 Visualizing a 3×3 singular symmetric matrix with repeated eigenvalues

The final example is a singular symmetric matrix with a repeated eigenvalue:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = 2$. One possible set of corresponding eigenvectors is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

In GeoGebra 3D, enter `A = {{1, 1, 0}, {1, 1, 0}, {0, 0, 0}}`, followed by `v_1 = Vector((1, -1, 0))`, `v_2 = Vector((0, 0, 1))`, and `v_3 = Vector((1, 1, 0))`. Then construct the image vectors with `w_1 = Vector(A v_1)`, `w_2 = Vector(A v_2)`, and `w_3 = Vector(A v_3)`.

Thus, the zero-eigenspace is a plane. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ shown in Figure 5 are not the only eigenvectors for $\lambda = 0$; rather, they form one convenient basis for this plane. Every vector in the plane $x + y = 0$ is mapped by A to the zero vector. This gives the visual meaning of the repeated zero eigenvalue: an entire plane collapses to the origin.

The nonzero eigenvalue has eigenspace

$$E_2 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

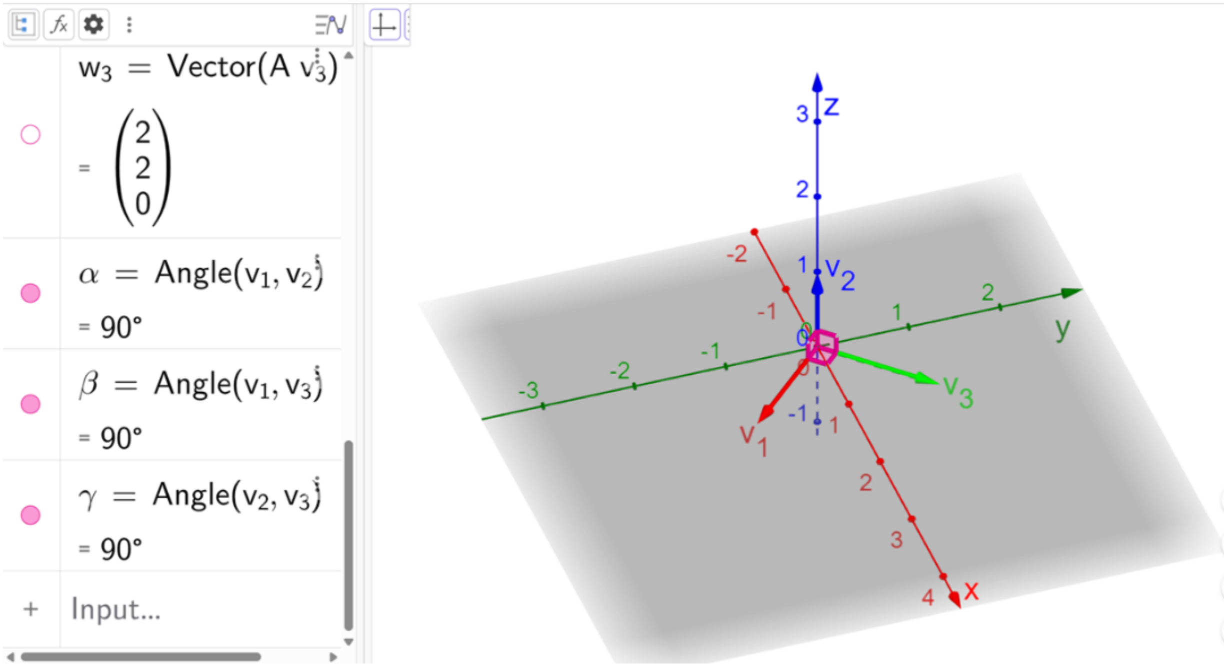

Figure 6. Visualizing the *symmetric matrix orthogonal eigenvectors* theorem.

Vectors on this line are stretched by a factor of 2, since $A\mathbf{v}_3 = 2\mathbf{v}_3$. The line E_2 is perpendicular to the plane E_0, illustrating the orthogonality of eigenspaces corresponding to distinct eigenvalues of a symmetric matrix.

Figure 6 shows that the selected basis vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are mutually orthogonal. This observation should be interpreted carefully. The symmetric matrix theorem guarantees orthogonality between eigenspaces associated with distinct eigenvalues, so E_0 is orthogonal to E_2. However, because $\mathbf{v}_1$ and $\mathbf{v}_2$ belong to the same eigenspace, their orthogonality is a consequence of the particular basis chosen for E_0, not a requirement imposed by the theorem. This distinction is pedagogically important because it helps students understand that a repeated eigenvalue corresponds to an eigenspace that may contain infinitely many possible eigenvectors and many possible bases.

A productive classroom activity is to ask students to create another vector in the plane $x + y = 0$, such as $\mathbf{u} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, and then construct $A\mathbf{u}$. Students should observe that $A\mathbf{u} = \mathbf{0}$, confirming that the entire plane is collapsed. They can then compare this behavior with vectors on the line E_2, which are stretched rather than collapsed.

7 Classroom use and extensions

The four examples can be used as a sequence of short exploratory tasks in a linear algebra course. Instructors may first ask students to compute eigenvalues and eigenvectors symbolically, then use GeoGebra to interpret the results dynamically. The goal is not to replace symbolic computation, but to help students coordinate the computation with the geometric behavior of the matrix transformation.

Several classroom prompts can be used across the examples:

- Which vectors keep their direction after multiplication by A? Which vectors reverse direction?
- How does the sign of an eigenvalue affect the direction of the transformed vector?
- What visual evidence shows that a vector belongs to the nullspace?
- In the repeated-eigenvalue case, why does an entire plane collapse to the origin rather than only the two displayed eigenvectors?
- Are all vectors inside the same repeated-eigenvalue eigenspace orthogonal to each other? How can GeoGebra be used to test this?
- What changes if one entry of the matrix is perturbed while symmetry is preserved?

These prompts are intended to move students from confirmation toward inquiry. For example, in the singular cases, students can be asked to explain why the transformed vector disappears rather than simply noting that the eigenvalue is zero. In the repeated-eigenvalue case, students can construct additional vectors in the zero-eigenspace and test whether they are mapped to the origin. As an extension, students can slightly change entries of a symmetric matrix and observe how the eigendirections and eigenvalues change. Such perturbation tasks encourage students to see eigenvalues and eigenvectors as objects connected to the structure of the matrix, not merely as outputs of a calculation.

8 Conclusion

This note has revised the visualization of symmetric matrices from a sequence of static confirmations into a set of dynamic, inquiry-oriented GeoGebra constructions. Across the four examples, students can compare each eigenvector $\mathbf{v}$ with its image $A\mathbf{v}$ and interpret the eigenvalue as the scalar that determines stretching, reversal, or collapse.

The two-dimensional examples introduce the basic geometric meaning of the eigenvector equation and the role of the zero eigenvalue in identifying a nullspace. The three-dimensional examples extend these ideas to an orthogonal eigenbasis and to a repeated-eigenvalue case. The repeated-eigenvalue example is especially important because it shows that an eigenvalue should be associated with an eigenspace rather than with a single displayed vector. In this case, the zero-eigenspace is a plane that collapses to the origin, while the eigenspace associated with the nonzero eigenvalue is a perpendicular line that is stretched by a factor of 2.

Overall, the GeoGebra constructions support students in coordinating algebraic calculations, geometric interpretation, and dynamic visual evidence. The constructions make visible several core features of symmetric matrices: real eigenvalues, orthogonal eigenspaces associated with distinct eigenvalues, invariant eigendirections, and the geometric meaning of singularity. These visualizations can therefore serve as classroom tools for helping students move beyond procedural calculation toward a more conceptual understanding of eigenvalues, eigenvectors, and eigenspaces.

9 References

Dorier, J.-L., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton, M. Artigue, U. Kirchgräber, J. Hillel, M. Niss, & A. Schoenfeld (Eds.), *The teaching and learning of mathematics at university level* (New ICMI Study Series, Vol. 7). Springer, Dordrecht. `https://doi.org/10.1007/0-306-47231-7_24`

Harel, G. (2000). Three principles of learning and teaching mathematics. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (Mathematics Education Library, Vol. 23). Springer, Dordrecht. `https://doi.org/10.1007/0-306-47224-4_6`

Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (Mathematics Education Library, Vol. 23). Springer, Dordrecht. `https://doi.org/10.1007/0-306-47224-4_7`